# 8 Applications of the Integrals

## Fastrack Revision

- If $f(x) \geq 0$ is a continuous function which is defined in the interval $[a, b]$, then the area of the region bounded by the curve $y = f(x)$, $X$-axis, $x = a$ and $x = b$ is given by

$$\text{Area} = \int_a^b f(x)\,dx = \int_a^b y\,dx$$

Here, curve lies above $X$-axis.

- If $f(x) \leq 0$ is a continuous function which is defined in the interval $[a, b]$, then area of the region bounded by the curve $y = f(x)$, $X$-axis, $x = a$ and $x = b$ is given by

$$\text{Area} = \left|\int_a^b y\,dx\right| = \left|\int_a^b f(x)\,dx\right| \quad \text{(take numerical value)}$$

Here, curve lies below $X$-axis.

- Area of the region bounded by the curve $x = f(y)$, $Y$-axis, $y = c$ and $y = d$ is given by

$$\text{Area} = \int_c^d x\,dy = \int_c^d f(y)\,dy$$

- If the curve $x = f(y)$, lies on left of $Y$-axis, then area of the region bounded by curve $x = f(y)$, $Y$-axis, $y = c$ and $y = d$ is given by

$$\text{Area} = \left|\int_c^d x\,dy\right| = \left|\int_c^d f(y)\,dy\right| \quad \text{(take numerical value)}$$

- If for $x \in [a, c]$, $f(x) \geq 0$ and for $x \in [c, b]$, $f(x) \leq 0$, where $a < c < b$, then area of region bounded by curve $y = f(x)$, $X$-axis, $x = a$ and $x = b$ is given by

$$\text{Area} = \int_a^c f(x)\,dx - \int_c^b f(x)\,dx$$

or $$\text{Area} = \int_a^c f(x)\,dx + \left|\int_c^b f(x)\,dx\right|$$

## Practice Exercise

### Multiple Choice Questions

**Q 1. Area enclosed by the circle $x^2 + y^2 = a^2$ is equal to:**

a. $2\pi a^2$ sq. units  b. $\pi a^2$ sq. units
c. $2\pi a$ sq. units  d. $\pi a$ sq. units

**Q 2. The area of the region bounded by the circle $x^2 + y^2 = 1$ is:** *(NCERT EXEMPLAR)*

a. $2\pi$ sq. units  b. $\pi$ sq. units
c. $3\pi$ sq. units  d. $4\pi$ sq. units

**Q 3. The area enclosed between the curve $x^2 + y^2 = 16$ and the coordinate axes in the first quadrant is:**

a. $4\pi$ sq. units  b. $3\pi$ sq. units
c. $2\pi$ sq. units  d. $\pi$ sq. units

**Q 4. The area bounded by the curve $x^2 + y^2 = 1$ in first quadrant is:**

a. $\frac{\pi}{4}$ sq. units  b. $\frac{\pi}{2}$ sq. units
c. $\frac{\pi}{3}$ sq. units  d. $\frac{\pi}{6}$ sq. units

**Q 5. Area lying in the first quadrant and bounded by the circle $x^2 + y^2 = 4$ and the lines $x = 0$ and $x = 2$ is:** *(NCERT EXERCISE)*

a. $\pi$ sq. units  b. $\frac{\pi}{2}$ sq. units
c. $\frac{\pi}{3}$ sq. units  d. $\frac{\pi}{4}$ sq. units

**Q 6. The area of the ellipse $\frac{x^2}{4^2} + \frac{y^2}{9^2} = 1$ is:**

a. $6\pi$ sq. units  b. $\frac{\pi(4^2 + 9^2)}{4}$ sq. units
c. $\pi(4 + 9)$ sq. units  d. None of these

**Q 7. The area of the region bounded by the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ is:**

a. $6\pi$ sq. units  b. $20\pi^2$ sq. units
c. $16\pi^2$ sq. units  d. $25\pi$ sq. units

**Q 8. The area bounded by the curve $2x^2 + y^2 = 2$ is:**

a. $\pi$ sq. units  b. $\sqrt{2}\pi$ sq. units
c. $\frac{\pi}{2}$ sq. units  d. $2\pi$ sq. units

**Q 9. Area bounded by the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$ is:**

a. $6\pi$ sq. units  b. $3\pi$ sq. units
c. $12\pi$ sq. units  d. None of these

**Q 10. The area enclosed by the curve $\frac{x^2}{25} + \frac{y^2}{9} = 1$ in first quadrant is:**

a. $10\pi$ sq. units  b. $\frac{15\pi}{4}$ sq. units
c. $5\pi$ sq. units  d. $4\pi$ sq. units

**Q 11. Area of the region bounded by the curve $y = x^2$ and the line $y = 4$ is:**

a. $\frac{11}{3}$ sq. units  b. $\frac{32}{3}$ sq. units
c. $\frac{43}{3}$ sq. units  d. $\frac{47}{3}$ sq. units

**Q 12.** Area of the region bounded by the curve $y^2 = 4x$, $Y$-axis and the line $y = 3$ is: *(NCERT EXERCISE)*

a. 2 sq. units
b. $\frac{9}{4}$ sq. units
c. $\frac{9}{3}$ sq. units
d. $\frac{9}{2}$ sq. units

**Q 13.** Area lying between the parabola $y^2 = 4x$ and the latus rectum is:

a. $\frac{1}{3}$ sq. units
b. $\frac{2}{3}$ sq. units
c. $\frac{5}{3}$ sq. units
d. $\frac{8}{3}$ sq. units

**Q 14.** The area bounded by the curve $y^2 = x$, line $y = 4$ and $Y$-axis is:

a. $\frac{16}{3}$ sq. units
b. $\frac{64}{3}$ sq. units
c. $7\sqrt{2}$ sq. units
d. None of these

**Q 15.** The area bounded by the curve $x = 3y^2 - 9$ and the lines $x = 0$, $y = 0$ and $y = 1$ is:

a. 8. sq. units
b. 8/3 sq. units
c. 3/8 sq. units
d. 3 sq. units

**Q 16.** The area bounded by the curve $y^2 = 9x$ and the lines $x = 1$, $x = 4$ and $y = 0$ in the first quadrant is:

a. 7 sq. units
b. 14 sq. units
c. 28 sq. units
d. 14/3 sq. units

**Q 17.** Area bounded by the curves $y = \sin x$, the line $x = 0$ and the line $x = \frac{\pi}{2}$, is equal to:

a. $\pi$ sq. units
b. 1 sq. unit
c. $\frac{\pi}{2}$ sq. units
d. 2 sq. units

**Q 18.** If the area of the region bounded by the lines $y = mx$, $x = 1$, $x = 2$ and $X$-axis, is 6 sq. units, then $m$ is equal to:

a. 3
b. 1
c. 2
d. 4

**Q 19.** The area in the positive quadrant enclosed by the circle $x^2 + y^2 = 4$, the line $x = y\sqrt{3}$ and $X$-axis, is: *(CBSE 2022 Term-2)*

a. $\frac{\pi}{2}$ sq. units.
b. $\frac{\pi}{4}$ sq. units
c. $\frac{\pi}{3}$ sq. units
d. $\pi$ sq. units

**Q 20.** The area enclosed by $y = 3x - 5$, $y = 0$, $x = 3$ and $x = 5$, is:

a. 12 sq. units
b. 13 sq. units
c. $13\frac{1}{2}$ sq. units
d. 14 sq. units

## Assertion & Reason Type Questions

**Directions (Q. Nos. 21-25):** *In the following questions, each question contains Assertion (A) and Reason (R). Each question has 4 choices (a), (b), (c) and (d) out of which only one is correct. The choices are:*

a. Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of Assertion (A)
b. Both Assertion (A) and Reason (R) are true but Reason (R) is not the correct explanation of Assertion (A)
c. Assertion (A) is true and Reason (R) is false
d. Assertion (A) is false and Reason (R) is true

**Q 21.** **Assertion (A):** The area of the region bounded by the curve $y^2 = 4x$ and the line $x = 3$ is $8\sqrt{3}$ sq. units.

**Reason (R):** If $f(x) \geq 0$ is a continuous function which is defined in the interval $[a, b]$, then the area of the region bounded by the curve $y = f(x)$, $X$-axis, $x = a$ and $x = b$ is given by

$$\text{Area} = \int_a^b f(x)\,dx = \int_a^b y\,dx$$

**Q 22.** **Assertion (A):** The area bounded by the parabola $y^2 = 4ax$ and the lines $x = a$ and $x = 4a$ is $\frac{56a^2}{3}$ sq. units.

**Reason (R):** Area of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $\pi ab$ sq. units.

**Q 23.** **Assertion (A):** The area enclosed by the curve $|x| + |y| = 2$ is 8 units.

**Reason (R):** $|x| + |y| = 2$ represents a square of side length $\sqrt{8}$ units.

**Q 24.** **Assertion (A):** The area bounded by the curve $y = 2\cos x$ and the $X$-axis from $x = 0$ to $x = 2\pi$ is 8 sq. units.

**Reason (R):** The area bounded by the curve $y = \sin x$ between $x = 0$ and $x = 2\pi$ is 2 sq. units.

**Q 25.** **Assertion (A):** The area of the region in the first quadrant, bounded by the parabola $y = 9x^2$ and the lines $x = 0$, $y = 1$ and $y = 4$ is 14/9 sq. units.

**Reason (R):** If for $x \in [a, c]$, $f(x) \geq 0$ and for $x \in [c, b]$, $f(x) \leq 0$, where $a < c < b$, then area of region bounded by curve $y = f(x)$, $X$-axis, $x = a$ and $x = b$ is given by

$$\text{Area} = \int_a^c f(x)\,dx - \int_c^b f(x)\,dx.$$

## Answers

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1. (b) | 2. (b) | 3. (a) | 4. (a) | 5. (a) | 6. (d) | 7. (a) | 8. (b) | 9. (a) | 10. (b) |
| 11. (b) | 12. (b) | 13. (d) | 14. (b) | 15. (a) | 16. (b) | 17. (b) | 18. (d) | 19. (c) | 20. (d) |
| 21. (a) | 22. (b) | 23. (a) | 24. (c) | 25. (b) | | | | | |

## Case Study Based Questions

### Case Study 1

A bridge connects two districts 50 feet apart. The arch on the bridge is in parabolic form. The highest point on the bridge is 5 feet above the road at the middle of the bridge as shown in the figure.

*Based on the above information, solve the following questions:*

**Q 1. The equation of the parabola designed on the bridge is:**

a. $y^2 = 125x$ b. $y^2 = -125x$

c. $x^2 = 125y$ d. $x^2 = -125y$

**Q 2. The value of the integral $\int_{-25}^{25} \frac{x^2}{125} dx$ is:**

a. $\frac{1000}{3}$ sq. units b. $\frac{250}{3}$ sq. units

c. 1200 sq. units d. 0 sq. units

**Q 3. The integrand of the integral $\int_{-25}^{25} x^2 \sin x \, dx$ is .................... function.**

a. an even b. an odd

c. Neither odd nor even d. None of these

**Q 4. The area formed by the curve $y^2 = 25x$, X-axis, $x = 4$ and $x = 9$ is:**

a. $\frac{100}{3}$ sq. units b. $\frac{110}{3}$ sq. units

c. $\frac{190}{3}$ sq. units d. $\frac{200}{3}$ sq. units

**Q 5. The area formed by the curve $x^2 = 125y$, Y-axis, $y = 16$ and $y = 25$ is:**

a. $\frac{610\sqrt{5}}{3}$ sq. units b. $\frac{1000}{3}$ sq. units

c. $\frac{4}{3}$ sq. units d. None of these

### Solutions

1. Since, the bridge is open downwards.
Therefore, the equation of the parabola on the bridge is $x^2 = -4ay$.
From the given options, the equation is of the form $x^2 = -125y$.
So, option (d) is correct.

2. Here integrand $f(x) = \frac{x^2}{125}$

Now, $f(-x) = \frac{(-x)^2}{125} = \frac{x^2}{125} = f(x)$

$\therefore$ $f(x)$ is an even function.

**TR!CK**

$$\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{if } f(-x) = f(x) \\ 0, & \text{if } f(-x) = -f(x) \end{cases}$$

Now, $\int_{-25}^{25} \frac{x^2}{125} dx = 2\int_0^{25} \frac{x^2}{125} dx = \frac{2}{125}\left[\frac{x^3}{3}\right]_0^{25}$

$= \frac{2}{125} \times \frac{1}{3}[(25)^3 - 0]$

$= \frac{2 \times 25 \times 25 \times 25}{125 \times 3} = \frac{250}{3}$ sq. units

So, option (b) is correct.

3. Here, integrand $f(x) = x^2 \sin x$

Now, $f(-x) = (-x)^2 \sin(-x)$

$= x^2(-\sin x) = -x^2 \sin x = -f(x)$

$\therefore$ $f(x)$ is an odd function.

So, option (b) is correct.

4. Equation of curve, $y^2 = 25x$

$\therefore$ Required area $= \int_{x=4}^{x=9} y\,dx = \int_4^9 \sqrt{25x}\,dx$

$= 5\int_4^9 x^{1/2}\,dx = 5\left[\frac{x^{3/2}}{3/2}\right]_4^9$

$= 5 \times \frac{2}{3}[(9)^{3/2} - (4)^{3/2}]$

$= \frac{10}{3}(27 - 8) = \frac{190}{3}$ sq. units

So, option (c) is correct.

5. Equation of curve, $x^2 = 125y$

$\therefore$ Required area $= \int_{y=16}^{y=25} x\,dy$

$= \int_{16}^{25} \sqrt{125y}\,dy = 5\sqrt{5}\left[\frac{y^{3/2}}{3/2}\right]_{16}^{25}$

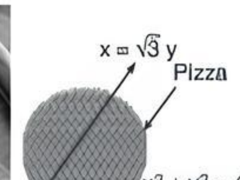
$= 5\sqrt{5} \times \frac{2}{3}[(25)^{3/2} - (16)^{3/2}]$

$= \frac{10\sqrt{5}}{3}(125 - 64) = \frac{610\sqrt{5}}{3}$ sq. units

So, option (a) is correct.

### Case Study 2

A man cut a pizza with a knife, pizza is circular in shape which is represented by $x^2 + y^2 = 4$ and sharp edge of knife represents a straight line given by $x = \sqrt{3}y$.



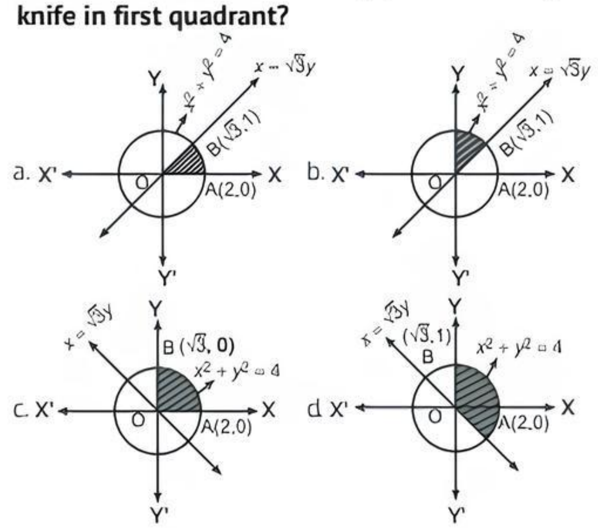
*Based on the given information, solve the following questions:*

**Q 1. The point(s) of intersection of the edge of knife (line) and pizza shown in the figure is (are):**

a. $(1, \sqrt{3}), (-1, -\sqrt{3})$ b. $(\sqrt{3}, 1), (-\sqrt{3}, -1)$

c. $(\sqrt{2}, 0), (0, \sqrt{3})$ d. $(-\sqrt{3}, 1), (1, -\sqrt{3})$

**Q 2. Which of the following shaded portion represent the smaller area bounded by pizza and edge of knife in first quadrant?**




**Q 3. Value of area of the region bounded by circular pizza and edge of knife in first quadrant is:**

a. $\frac{\pi}{2}$ sq. units b. $\frac{\pi}{3}$ sq. units

c. $\frac{\pi}{5}$ sq. units d. $\pi$ sq. units

**Q 4. Area of each slice of pizza when a man cut the pizza into 4 equal pieces, is:**

a. $\pi$ sq. units b. $\frac{\pi}{2}$ sq. units

c. $3\pi$ sq. units d. $2\pi$ sq. units

**Q 5. Area of whole pizza is:**

a. $3\pi$ sq. units b. $2\pi$ sq. units

c. $5\pi$ sq. units d. $4\pi$ sq. units

## Solutions

1. We have, $x^2 + y^2 = 4$ ...(1)

and $x = \sqrt{3}y$ ...(2)

From eqs. (1) and (2), we get

$3y^2 + y^2 = 4 \Rightarrow 4y^2 = 4$

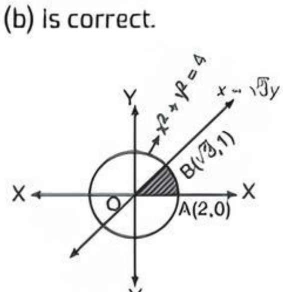
$\Rightarrow y^2 = 1 \Rightarrow y = \pm 1$

From eq. (2), $x = \sqrt{3}, -\sqrt{3}$

$\therefore$ Points of intersection of pizza and edge of knife are $(\sqrt{3}, 1), (-\sqrt{3}, -1)$.

So, option (b) is correct.

2.




So, option (a) is correct.

3. Required area $= \int_0^{\sqrt{3}} \frac{x}{\sqrt{3}} dx + \int_{\sqrt{3}}^{2} \sqrt{4 - x^2}\, dx$

$$= \frac{1}{\sqrt{3}}\left[\frac{x^2}{2}\right]_0^{\sqrt{3}} + \left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right)\right]_{\sqrt{3}}^{2}$$

$$= \frac{1}{\sqrt{3}}\left[\frac{3}{2} - 0\right] + \left[2\sin^{-1}(1) - \left(\frac{\sqrt{3}}{2} + 2\sin^{-1}\frac{\sqrt{3}}{2}\right)\right]$$

$$= \frac{\sqrt{3}}{2} + \frac{2\pi}{2} - \frac{\sqrt{3}}{2} - \frac{2\pi}{3} = \frac{\pi}{3} \text{ sq. units}$$

So, option (b) is correct.

4. We have, $x^2 + y^2 = 4$

$\Rightarrow (x - 0)^2 + (y - 0)^2 = (2)^2$

$\therefore$ Radius $= 2$

Area of $\frac{1}{4}$th slice of pizza $= \frac{1}{4}\pi(2)^2 = \pi$ sq. units

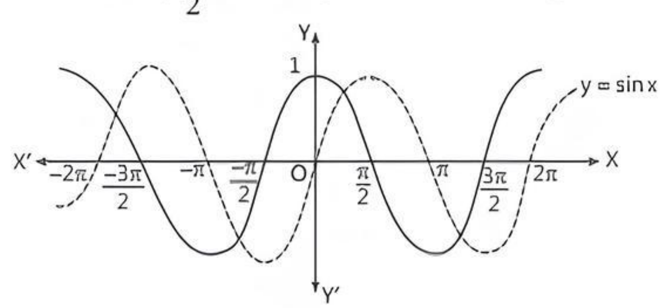
So, option (a) is correct.

5. Area of whole pizza $= \pi(2)^2 = 4\pi$ sq. units

So, option (d) is correct.

## Case Study 3

In a classroom, teacher explains the properties of a particular curve by saying that this particular curve has beautiful ups and downs. It starts at 1 and heads down until $\pi$ radian and then heads up again and closely related to sine function and both follow each other, exactly $\frac{\pi}{2}$ radians apart as shown in figure.




*Based on the above information, solve the following questions:*

**Q 1. Write the name of curve, about which teacher explained in the classroom.**

**Q 2. Find the area of curve explained in the above passage from 0 to $\frac{\pi}{2}$.**

**Q 3. Find the area of curve discussed in the above passage from $\frac{\pi}{2}$ to $\frac{3\pi}{2}$.**

***Or***

**Find the area of curve discussed in the above passage from $\frac{3\pi}{2}$ to $2\pi$.**

## Solutions

1. Here, teacher explained about cosine curve.

2. Required area $= \int_0^{\pi/2} \cos x\, dx$

$$= [\sin x]_0^{\pi/2} = \sin\frac{\pi}{2} - \sin 0 = 1 - 0 = 1 \text{ sq. unit}$$

3. Required area $= \left| \int_{\pi/2}^{3\pi/2} \cos x \, dx \right|$

$= |[\sin x]_{\pi/2}^{3\pi/2}|$

$= \left| \sin\frac{3\pi}{2} - \sin\frac{\pi}{2} \right|$

$= |-1-1| = |-2| = 2$ sq. units

[Since, area can't be negative]

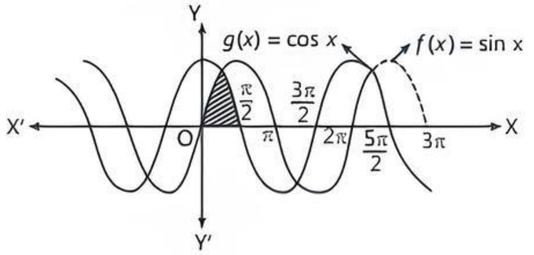
*Or*

Required area $= \int_{3\pi/2}^{2\pi} \cos x \, dx = [\sin x]_{3\pi/2}^{2\pi}$

$= \sin 2\pi - \sin\frac{3\pi}{2} = 0-(-1) = 1$ sq. unit

## Case Study 4

Graphs of two functions $f(x) = \sin x$ and $g(x) = \cos x$ is given below:




*Based on the above information, solve the following questions:*

**Q 1. In [0, π], the curves $f(x) = \sin x$ and $g(x) = \cos x$ intersect at $x =$**

a. $\frac{\pi}{2}$ b. $\frac{\pi}{3}$ c. $\frac{\pi}{4}$ d. $\pi$

**Q 2. The value of $\int_0^{\pi/4} \sin x \, dx$ is:**

a. $1-\frac{1}{\sqrt{2}}$ b. $1+\frac{1}{\sqrt{2}}$ c. $2-\frac{1}{\sqrt{2}}$ d. $2+\frac{1}{\sqrt{2}}$

**Q 3. The value of $\int_{\pi/4}^{\pi/2} \cos x \, dx$ is:**

a. $1+\frac{1}{\sqrt{2}}$ b. $1-\frac{1}{\sqrt{2}}$ c. $2-\sqrt{2}$ d. $2+\sqrt{2}$

**Q 4. The value of $\int_0^{\pi} \sin x \, dx$ is:**

a. 0 b. 1

c. 2 d. −2

**Q 5. The value of $\int_0^{\pi/2} \sin x \, dx$ is:**

a. 0 b. 1

c. 2 d. −2

### Solutions

1. For point of intersection, we have $\sin x = \cos x$

$\Rightarrow \quad \frac{\sin x}{\cos x} = 1$

$\Rightarrow \quad \tan x = 1 \Rightarrow x = \frac{\pi}{4}$

So, option (c) is correct.

2. $\int_0^{\pi/4} \sin x \, dx = [-\cos x]_0^{\pi/4}$

$= -\cos\frac{\pi}{4} + \cos 0 = 1 - \frac{1}{\sqrt{2}}$

So, option (a) is correct.

3. $\int_{\pi/4}^{\pi/2} \cos x \, dx = [\sin x]_{\pi/4}^{\pi/2}$

$= \sin\frac{\pi}{2} - \sin\frac{\pi}{4} = 1 - \frac{1}{\sqrt{2}}$

So, option (b) is correct.

4. $\int_0^{\pi} \sin x \, dx = [-\cos x]_0^{\pi}$

$= [-\cos\pi + \cos 0] = (1+1) = 2$

So, option (c) is correct.

5. $\int_0^{\pi/2} \sin x \, dx = [-\cos x]_0^{\pi/2}$

$= \left[-\cos\frac{\pi}{2} + \cos 0\right] = 0+1 = 1$

So, option (b) is correct.

## Case Study 5

Consider the following equations of curves $x^2 = y$ and $y = x$.

*Based on the above information, solve the following questions:*

**Q 1. Find the point(s) of intersection of both the curves.**

**Q 2. Draw the graph of area bounded by the curves.**

**Q 3. Find the value of the integral $\int_0^1 x \, dx$.**

***Or***

**Find the value of the integral $\int_0^1 x^2 \, dx$.**

### Solutions

1. We have, $x^2 = y$ ...(1)

and $x = y$ ...(2)

From eqs. (1) and (2),

$x^2 = x$

$\Rightarrow \quad x^2 - x = 0$

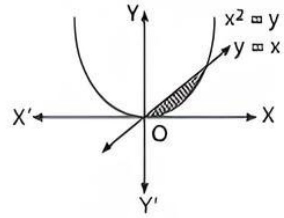
$\Rightarrow \quad x(x-1) = 0$

$\Rightarrow \quad x = 0, 1$

From eq. (2), $y = 0, 1$

Thus, required points of intersection are (0, 0), (1, 1).

2.




3. $\int_0^1 x \, dx = \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2} - 0 = \frac{1}{2}$

*Or*

$\int_0^1 x^2 \, dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3} - 0 = \frac{1}{3}$

## Case Study 6

Consider the following equation of curve $y^2 = 4x$ and straight line $x + y = 3$.

*Based on the above information, solve the following questions:*

**Q 1. Write the points where the line $x + y = 3$ cuts the $X$ and $Y$-axes.**

**Q 2. Find the point(s) of intersection of two given curves.**

**Q 3. Draw the graph and represent the shaded portion of area bounded by the given curves.**

***Or***

**Find the value of integral $\int_{-6}^{2} (3-y)\,dy$.**

### Solutions

1. Line $x + y = 3$ cuts the $X$-axis and $Y$-axis at (3, 0) and (0, 3) respectively.

[Since, at $X$-axis, $y = 0$ and at $Y$-axis, $x = 0$]

2. We have, $y^2 = 4x$ ...(1)

and $x + y = 3$ ...(2)

From eqs. (1) and (2), we have

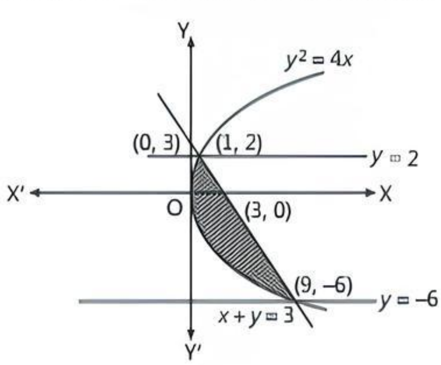
$$y^2 = 4(3-y)$$
$$\Rightarrow y^2 + 4y - 12 = 0$$
$$\Rightarrow y^2 + (6-2)y - 12 = 0$$
$$\Rightarrow y^2 + 6y - 2y - 12 = 0$$
$$\Rightarrow y(y+6) - 2(y+6) = 0$$
$$\Rightarrow (y+6)(y-2) = 0$$
$$\Rightarrow y = 2, \quad y = -6$$

From eq. (2), $x = 3 - 2 = 1$

or $x = 3 + 6 = 9$

$\therefore$ Required points of intersection are (1, 2), (9, –6).

3.




*Or*

$$\int_{-6}^{2} (3-y)\,dy = \left[3y - \frac{y^2}{2}\right]_{-6}^{2}$$
$$= \left[6 - \frac{4}{2}\right] - \left[3(-6) - \frac{(-6)^2}{2}\right]$$
$$= 4 + 36 = 40$$

## Case Study 7

A mirror in the shape of an ellipse is represented by $\frac{x^2}{9} + \frac{y^2}{4} = 1$ was hanging on the wall. Sanjeev and his daughter were playing with football inside the house, even his wife refused to do so. All of a sudden, football hit the mirror and got a scratch in the shape of line represented by $\frac{x}{3} + \frac{y}{2} = 1$.

*Based on the above information, solve the following questions:*

**Q 1. Find the point(s) of intersection of ellipse and scratch (straight line).**

**Q 2. Draw the graph and show the area of smaller region bounded by the ellipse and line.**

**Q 3. Find the value of $\frac{2}{3}\int_0^3 \sqrt{9-x^2}\,dx$.**

***Or***

**Find the value of $2\int_0^3 \left(1 - \frac{x}{3}\right) dx$.**

### Solutions

1. We have,

$$\frac{x^2}{9} + \frac{y^2}{4} = 1 \quad ...(1)$$

and $$\frac{x}{3} + \frac{y}{2} = 1 \quad ...(2)$$

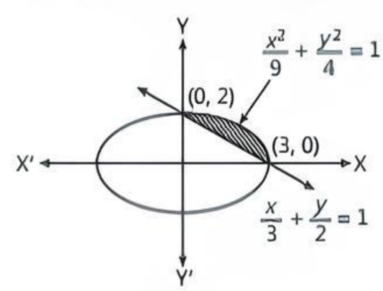
From eq. (1), we have

$$\frac{1}{9}\cdot x^2 + \frac{1}{4}\cdot\left\{2\left(1-\frac{x}{3}\right)\right\}^2 = 1$$
$$\Rightarrow \frac{x^2}{9} + 1 + \frac{x^2}{9} - \frac{2x}{3} = 1$$
$$\Rightarrow \frac{2x^2}{9} - \frac{2x}{3} = 0$$
$$\Rightarrow \frac{2x}{3}\left(\frac{x}{3} - 1\right) = 0$$
$$\Rightarrow x = 0, 3$$

From eq. (2); $y = 2, 0$

$\therefore$ Required points of intersection are (0, 2) and (3, 0).

2.




3. $$\frac{2}{3}\int_0^3 \sqrt{9-x^2}\,dx = \frac{2}{3}\int_0^3 \sqrt{(3)^2 - x^2}\,dx$$
$$= \frac{2}{3}\left[\frac{x}{2}\sqrt{9-x^2} + \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right)\right]_0^3$$

$$= \frac{2}{3}\left[\frac{3}{2}(0) + \frac{9}{2}\sin^{-1}(1) - \frac{1}{2}(0) - \frac{9}{2}\sin^{-1}(0)\right]$$

$$= \frac{2}{3}\left[\frac{9}{2}\cdot\frac{\pi}{2}\right] = \frac{3\pi}{2}$$

*Or*

$$2\int_0^3 \left(1 - \frac{x}{3}\right) dx = 2\left[x - \frac{x^2}{6}\right]_0^3$$

$$= 2\left(3 - \frac{9}{6} - 0 - 0\right) = 2 \times \frac{3}{2} = 3$$

## Very Short Answer Type Questions

Q 1. Find the area of the region bounded by the curve $y = x^2$ and the line $y = 4$. *(NCERT EXERCISE)*

Q 2. Find the area of the region bounded by the parabola $y^2 = 8x$ and the line $x = 2$. *(NCERT EXEMPLAR; CBSE 2020)*

Q 3. Using integration, find the area of the region bounded by lines $x - y + 1 = 0$, $x = -2$, $x = 3$ and $X$-axis. *(CBSE 2022 Term-2)*

Q 4. Find the area bounded by the curve $x = 2y + 3$, $Y$-axis and the lines $y = 1$ and $y = -1$. *(NCERT EXEMPLAR)*

Q 5. Find the area of the region enclosed by the curves $y^2 = x$, $x = \frac{1}{4}$, $y = 0$ and $x = 1$ using integration. *(CBSE 2022 Term-2)*

Q 6. Find the area of the region bounded by $y^2 = 9x$, $x = 2$, $x = 4$ and the $X$-axis in the first quadrant. *(NCERT EXERCISE)*

Q 7. Find the area bounded by $y = x^2$, $X$-axis and the lines $x = -1$ and $x = 1$.

Q 8. Find the area between 0 and $\pi$ of the curve $y = \sin x$. *(NCERT EXEMPLAR)*

Q 9. Find the area of the region bounded by the curve $y = \cos x$ between $x = 0$ and $x = \pi$. *(NCERT EXEMPLAR)*

Q 10. Find the area bounded by the line $y = x$, $X$-axis and the lines $x = -1$ to $x = 2$.

Q 11. The area bounded by the curve $y = f(x)$, the $X$-axis and $x = 1$ and $x = b$ is $(b - 1)\sin(3b + 4)$. Find $f(x)$.

## Short Answer Type-I Questions

Q 1. Find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ using integration method. *(NCERT EXERCISE, NCERT EXEMPLAR)*

Q 2. Find the area of the ellipse $x^2 + 9y^2 = 36$ using integration. *(CBSE 2020)*

Q 3. Sketch the region bounded by the lines $2x + y = 8$, $y = 2$, $y = 4$ and the $Y$-axis. Hence, obtain its area using integration. *(CBSE 2023)*

Q 4. Find the area of the region bounded by the line $y = 3x + 2$, $X$-axis and the ordinates $x = -1$ and $x = 1$. *(NCERT EXERCISE)*

## Short Answer Type-II Questions

Q 1. Find the area bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the ordinates $x = 0$ and $x = ae$. *(NCERT EXERCISE)*

Q 2. Find the area bounded by the curve $y = \cos x$ between $x = 0$ and $x = 2\pi$.

Q 3. Find the area of the region bounded by $y = |x|$, $x \le 5$ in the first quadrant.

Q 4. Using integration, find the area of the region bounded by $y = mx$ $(m > 0)$, $x = 1$, $x = 2$ and the $X$-axis. *(CBSE 2023)*

Q 5. Find the area of the region bounded by the parabola $y^2 = 4ax$ and the straight line $y = mx$. *(NCERT EXERCISE)*

Q 6. Find the area bounded by the lines $y = |x - 2|$, $x = 1$, $x = 3$ and the $X$-axis.

Q 7. Find the area of the minor segment of the circle $x^2 + y^2 = 4$ cut-off by the line $x = 1$, using integration. *(CBSE 2023)*

Q 8. Find the area bounded by the curve $y = \cos x$ between $x = 0$ and $x = \frac{3\pi}{2}$.

Q 9. Find the area of the following region using integration:
$\{(x, y) : y^2 \le 2x \text{ and } y \ge x - 4\}$ *(CBSE 2023)*

## Long Answer Type Questions

Q 1. Using integration, find the area of the region bounded by the parabola $y^2 = 4ax$ and its latus rectum. *(NCERT EXERCISE; CBSE 2023)*

Q 2. Find the area of minor portion of the curve $9x^2 + 16y^2 = 144$ intercepted by the line $x = 2$ using definite integration.

Q 3. Using integration, find the area of the region in the first quadrant enclosed by the line $x + y = 2$, the parabola $y^2 = x$ and the $X$-axis. *(NCERT; CBSE SQP 2022 Term-2)*

Q 4. Using integration, find the area of the region in the first quadrant enclosed by the $X$-axis, the line $y = x$ and the circle $x^2 + y^2 = 32$. *(NCERT EXERCISE, CBSE 2018)*

Q 5. Using integration, find the area of the region bounded by the circle $x^2 + y^2 = 16$, line $y = x$ and $Y$-axis, but lying in the 1st quadrant. *(CBSE 2023)*

**Q 6.** Find the area of triangle whose two vertices formed from the $X$-axis and the line $y = 3 - |x|$.

**Q 7.** Find the ratio of the areas of two parts of the circle $x^2 + y^2 = a^2$ divided by the line $x = \frac{1}{2}a$.

*(NCERT EXEMPLAR)*

**Q 8.** Find the area of the region bounded by the curve $y = \tan x$, line $x = \frac{\pi}{4}$ and the $X$-axis.

**Q 9.** Using integration, find the area of the region

$$\{(x, y) : 0 \le y \le \sqrt{3}x, x^2 + y^2 \le 4\}$$

*(CBSE SQP 2022 Term-2)*

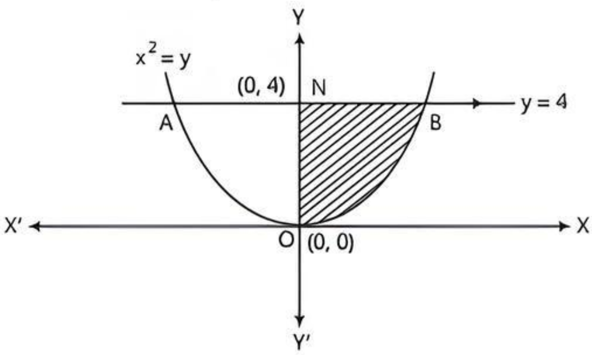
**Q 10.** Make a rough sketch of the region $\{(x, y) : 0 \le y \le x^2 + 1, 0 \le y \le x + 1, 0 \le x \le 2\}$ and find the area of the region, using the method of integration. *(CBSE SQP 2023-24)*

**Q 11.** Find the area bounded by the curve $x = 0$ and $x + 2|y| = 1$.

## Solutions

### Very Short Answer Type Questions

**1.** Curve represented by the given equation $y = x^2$ is a parabola, which is symmetrical about $Y$-axis.




**TiP**

*Do practice for taking limits of y.*

$\therefore$ Required area of region $AOBA$

$$= 2 \times \text{Area of shaded region} = 2\int_0^4 x\, dy$$

$$= 2\int_0^4 \sqrt{y}\, dy$$

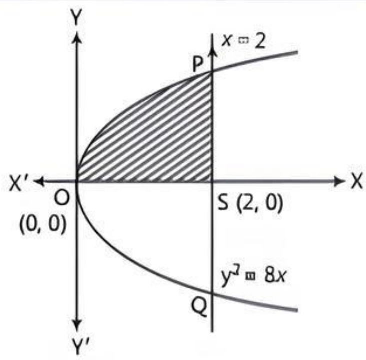
$$= 2\left[\frac{2}{3}y^{3/2}\right]_0^4 = \frac{4}{3}[(4)^{3/2} - 0]$$

$$= \frac{4}{3} \times 8 = \frac{32}{3} \text{ sq. units}$$

**2.** Let the line $x = 2$ meets the parabola at points $P$ and $Q$.

**TR!CK**

*Curve is symmetrical about X-axis i.e., area of both portions are equal numerically.*




We have to find the area of region $POQP$ which is double the area of shaded region $PSOP$.

$\because$ Required area $= 2 \times$ Area of shaded region

$$= 2\int_0^2 y\, dx = 2\int_0^2 \sqrt{8x}\, dx$$

$$= 4\sqrt{2}\int_0^2 \sqrt{x}\, dx = 4\sqrt{2}\left[\frac{2}{3}x^{3/2}\right]_0^2$$

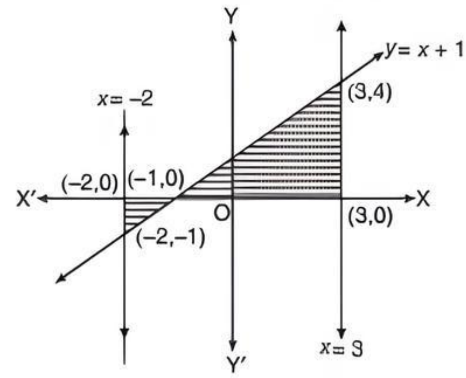
$$= \frac{8\sqrt{2}}{3}(2^{3/2} - 0) = \frac{8\sqrt{2}}{3} \times 2\sqrt{2}$$

$$= \frac{32}{3} \text{ sq. units}$$

**3.** $\therefore$ Required area

= Area of shaded region.

$$= \left|\int_{-2}^{-1}(x+1)dx\right| + \int_{-1}^{3}(x+1)\, dx$$




$$= \left|\left[\frac{x^2}{2} + x\right]_{-2}^{-1}\right| + \left[\frac{x^2}{2} + x\right]_{-1}^{3}$$

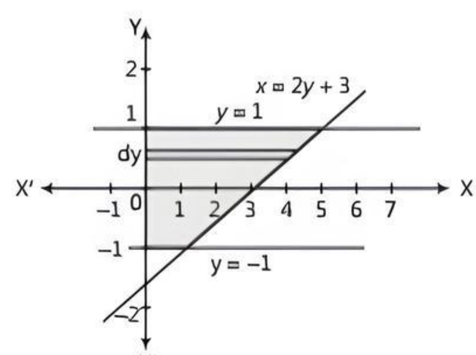
$$= \left|\frac{1}{2} - 1 - \left(\frac{4}{2} - 2\right)\right| + \left\{\frac{9}{2} + 3 - \frac{1}{2} + 1\right\}$$

$$= \left|\frac{-1}{2} - 0\right| + (8) = \frac{1}{2} + 8 = \frac{17}{2} \text{ sq. units}$$

**4.**

**TiP**

*Learn to draw the graphs correctly.*



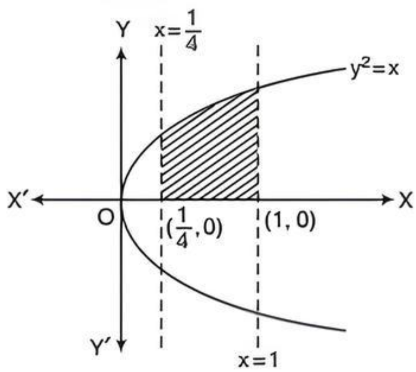
From the figure required area of the shaded region

$$= \int_{-1}^{1} (2y+3)\,dy = [y^2+3y]_{-1}^{1}$$

$$= [1+3-1+3] = 6 \text{ sq. units.}$$

**5.** We have equation of parabola,

$$y^2 = x$$




$\therefore$ Required area $= \int_{1/4}^{1} \sqrt{x}\,dx = \left[\frac{2}{3}x^{3/2}\right]_{1/4}^{1}$

$$= \frac{2}{3}\left[(1)^{3/2} - \left(\frac{1}{4}\right)^{3/2}\right]$$

$$= \frac{2}{3}\left(1-\frac{1}{8}\right) = \frac{2}{3}\times\frac{7}{8}$$

$$= \frac{7}{12} \text{ sq. units}$$

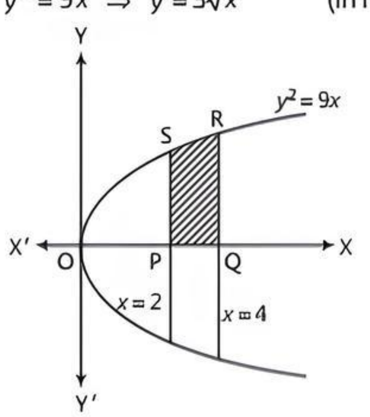
**6.** The curve $y^2 = 9x$ is a parabola whose vertex is (0, 0).

Curve is symmetric about $X$-axis.

Region $PQRS$ is bounded by the curve

$$y^2 = 9x, \quad x = 2, \quad x = 4 \quad \text{and} \quad X\text{-axis.}$$

Now, $y^2 = 9x \Rightarrow y = 3\sqrt{x}$ (in first quadrant)




$\therefore$ Area of region $PQRS = \int_{2}^{4} 3\sqrt{x}\,dx$

$$= 3\times\left[\frac{x^{3/2}}{3/2}\right]_2^4 = 3\times\frac{2}{3}\left(x^{3/2}\right)_2^4$$

$$= 2\left(4^{3/2} - 2^{3/2}\right)$$

$$= 2\left(8 - 2\sqrt{2}\right)$$

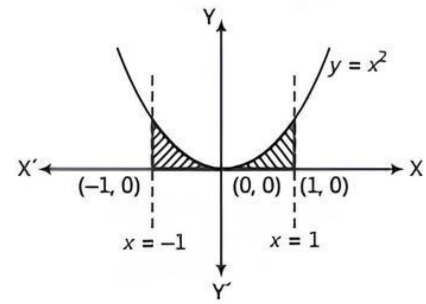
$$= 16 - 4\sqrt{2} \text{ sq. units}$$

**COMMON ERROR**

*Some students fail to draw the correct graph of parabola.*

**7.** From the figure, required area of the shaded region

$$= \int_{-1}^{1} y\,dx = \int_{-1}^{1} x^2\,dx$$




**TR!CK**

$$\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{if } f(-x) = f(x) \\ 0, & \text{if } f(-x) = -f(x) \end{cases}$$

Here, integrand $f(x) = x^2$

$\therefore$ $f(-x) = (-x)^2 = x^2 = f(x)$

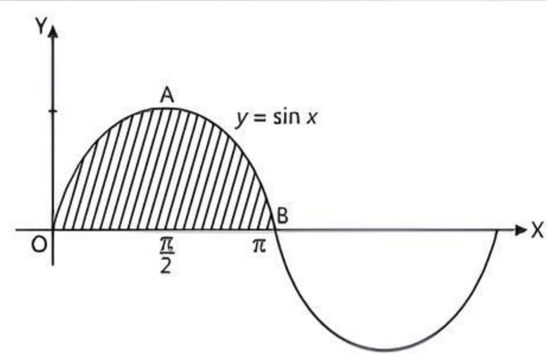
$$= 2\int_0^1 x^2\,dx = 2\left[\frac{x^3}{3}\right]_0^1$$

$$= \frac{2}{3}(1-0) = \frac{2}{3} \text{ sq. units}$$

**8.**

**TiP**

*Learn to draw the graphs of trigonometric functions correctly.*

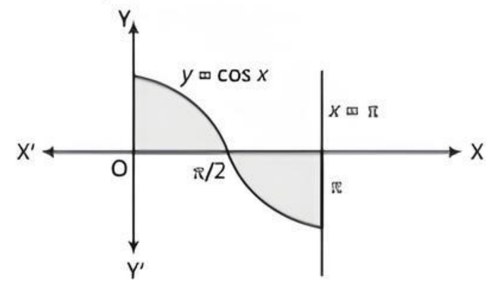



$\therefore$ Area of curve OAB $= \int_0^{\pi} y\,dx = \int_0^{\pi} \sin x\,dx = [-\cos x]_0^{\pi}$

$$= \cos 0 - \cos \pi$$

$$= 1 - (-1) = 2 \text{ sq. units}$$

**9.** We have, $y = \cos x$, $x = 0$, $x = \pi$




**TiP**

*Learn the intervals where graph of cos x is positive and negative.*

From the figure, required area of the shaded region

$$= \int_0^{\pi} |\cos x|\, dx$$

$$= \int_0^{\pi/2} \cos x\, dx + \int_{\pi/2}^{\pi} (-\cos x)\, dx$$

$$= [\sin x]_0^{\pi/2} - [\sin x]_{\pi/2}^{\pi}$$

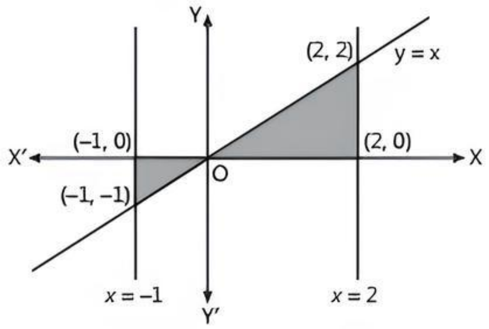
$$= \left[\sin\frac{\pi}{2} - \sin 0\right] - \left[\sin \pi - \sin\frac{\pi}{2}\right]$$

$$= (1-0) - (0-1) = 2 \text{ sq. units}$$

**COMMON ERROR**

*Students fail to apply the limits correctly.*

**10.** We have, $y = x$




$\therefore$ Required area = area of shaded region

$$= \left|\int_{-1}^{0} x\, dx\right| + \left|\int_0^2 x\, dx\right|$$

$$= \left|\frac{x^2}{2}\right|_{-1}^{0} + \left|\frac{x^2}{2}\right|_0^2$$

$$= \left|-\frac{1}{2}\right| + |2| = 2 + \frac{1}{2}$$

$$= \frac{5}{2} \text{ sq. units}$$

**11.** Given, $\int_1^b f(x)\, dx = (b-1)\sin(3b+4)$

Area of function $= \int_1^x f(x)\, dx$

$$= (x-1)\sin(3x+4)$$

On differentiating, we get

$$f(x) = \sin(3x+4) + 3(x-1)\cdot\cos(3x+4)$$

## Short Answer Type-I Questions

**1.** Equation of ellipse: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ...(1)

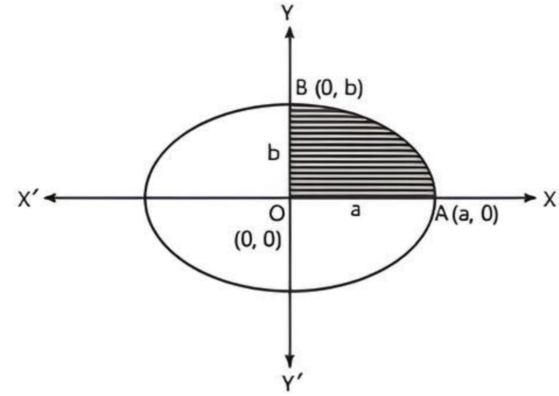
For ellipse, major axis $= 2a$,

Minor axis $= 2b$

and centre $= (0, 0)$

**TiP**

*Learn to sketch the graphs of circle, ellipse and parabola from a standard equation.*




$\because$ Ellipse is symmetrical about both axes.

$\therefore$ Required area = 4 × Area of shaded region

$$= 4\int_0^a y\, dx$$

$$= 4\int_0^a \frac{b}{a}\cdot\sqrt{a^2 - x^2}\, dx \qquad \text{[From eq. (1)]}$$

$$= \frac{4b}{a}\left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{1}{2}a^2\sin^{-1}\frac{x}{a}\right]_0^a$$

$$= \frac{4b}{a}\left[\frac{1}{2}a^2\sin^{-1}1 - 0\right]$$

$$= \frac{4b}{a} \times \frac{1}{2}a^2 \times \frac{\pi}{2} = \pi ab \text{ sq. units}$$

**2.** Given, $x^2 + 9y^2 = 36$

or $\frac{x^2}{36} + \frac{y^2}{4} = 1$

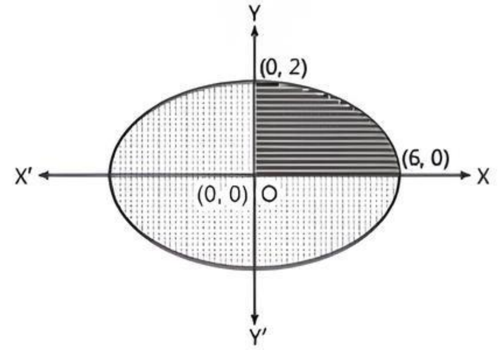
$\Rightarrow$ $a^2 = 36 \Rightarrow a = \pm 6$

and $b^2 = 4 \Rightarrow b = \pm 2$

Also, $9y^2 = 36 - x^2$

$\Rightarrow$ $y = \pm\frac{1}{3}\sqrt{36 - x^2}$

For first quadrant, $y = \frac{1}{3}\sqrt{36 - x^2}$




$\therefore$ Required area = 4 × Area of shaded region

$$= 4\int y\, dx = 4\int_0^6 \frac{1}{3}\sqrt{36 - x^2}\, dx$$

$$= \frac{4}{3}\int_0^6 \sqrt{(6)^2 - x^2}\, dx$$

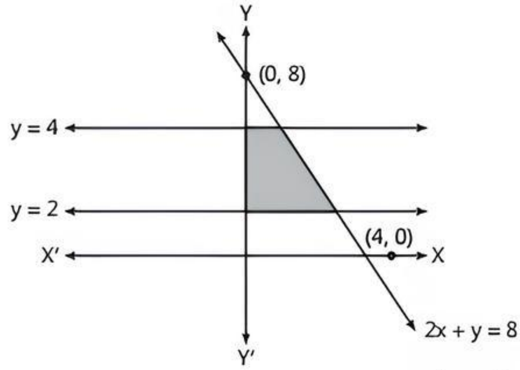
$$= \frac{4}{3}\left[\frac{x}{2}\sqrt{6^2 - x^2} + \frac{36}{2}\sin^{-1}\left(\frac{x}{6}\right)\right]_0^6$$

$$= \frac{4}{3}\left[18 \times \frac{\pi}{2} - 0\right] = 12\pi \text{ sq. units}$$

**COMMON ERROR**

*Some students fail to find the standard equation of the ellipse and hence get wrong figure.*

3. Given equation of line is $2x + y = 8$, which intercept $X$ and $Y$-axes at points (4, 0) and (0, 8).




$\therefore$ Area of bounded region $= \int_2^4 x\,dy = \int_2^4 \left(\frac{8-y}{2}\right) dy$

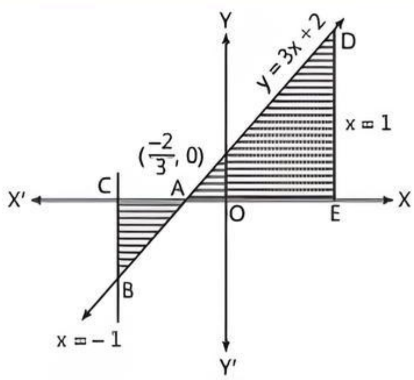
$$= \frac{1}{2}\left[8y - \frac{y^2}{2}\right]_2^4 = \frac{1}{2}[32 - 8 - (16 - 2)]$$

$$= \frac{1}{2}[24 - 14] = \frac{10}{2} = 5 \text{ sq. units}$$

4. From the figure, the line $y = 3x + 2$, meets $X$-axis at $x = -\frac{2}{3}$ and its graph below the $X$-axis for $x \in \left(-1, -\frac{2}{3}\right)$ and above the $X$-axis for $x \in \left(-\frac{2}{3}, 1\right)$.

**TR!CK**

$\int_a^c f(x)dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$, *where* $a < b < c$.




$\therefore$ Required area = Area of the region ACBA + Area of the region ADEA

$$= \left|\int_{-1}^{-2/3} (3x+2)\,dx\right| + \int_{-2/3}^{1} (3x+2)\,dx$$

$$= \left|\left[\frac{3x^2}{2} + 2x\right]_{-1}^{-2/3}\right| + \left[\frac{3x^2}{2} + 2x\right]_{-2/3}^{1}$$

$$= \left|\frac{2}{3} - \frac{4}{3} - \left(\frac{3}{2} - 2\right)\right| + \left(\frac{3}{2} + 2\right) - \left(\frac{2}{3} - \frac{4}{3}\right)$$

$$= \left|\frac{-2}{3} + \frac{1}{2}\right| + \frac{7}{2} + \frac{2}{3} = \left|\frac{-1}{6}\right| + \frac{25}{6}$$

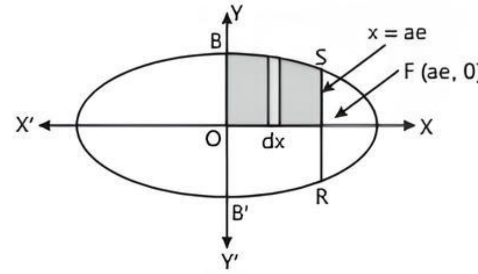
$$= \frac{1}{6} + \frac{25}{6} = \frac{26}{6} = \frac{13}{3} \text{ sq. units}$$

**COMMON ERR!OR**

*Some students fail to break the limits according to the figure.*

## Short Answer Type-II Questions

1. The required area of the region BOB′RFSB is enclosed by the ellipse and the lines $x = 0$ and $x = ae$.




$\therefore$ Area of the region BOB′RFSB $= 2\int_0^{ae} y\,dx$

$$= 2\frac{b}{a}\int_0^{ae} \sqrt{a^2 - x^2}\,dx$$

$$= \frac{2b}{a}\left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^{ae}$$

$$= \frac{2b}{2a}\left[ae\sqrt{a^2 - a^2e^2} + a^2\sin^{-1} e\right]$$

$$= \frac{b}{a}\left[a^2e\sqrt{1-e^2} + a^2\sin^{-1} e\right]$$

$$= ab\left[e\sqrt{1-e^2} + \sin^{-1} e\right] \text{ sq. units}$$

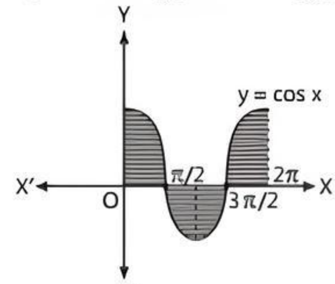
2.

**TiP**

*Learn to draw the graph of cos x and understand where the graph is positive and negative.*

$\therefore$ Required area $= \int_0^{\pi/2} \cos x\,dx + \int_{\pi/2}^{3\pi/2} (-\cos x)\,dx + \int_{3\pi/2}^{2\pi} \cos x\,dx$

$$= [\sin x]_0^{\pi/2} - [\sin x]_{\pi/2}^{3\pi/2} + [\sin x]_{3\pi/2}^{2\pi}$$

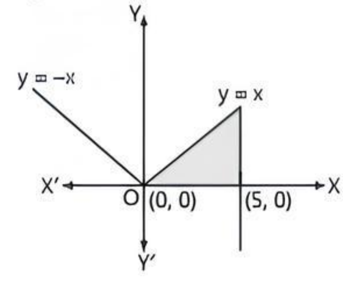



$$= \left[\sin\frac{\pi}{2} - \sin 0\right] - \left[\sin\frac{3\pi}{2} - \sin\frac{\pi}{2}\right] + \left[\sin 2\pi - \sin\frac{3\pi}{2}\right]$$

$$= (1-0) - (-1-1) + (0+1) = 1 + 2 + 1 = 4 \text{ sq. units}$$

3. We have, $y = -x$, if $x < 0$ ...(1)

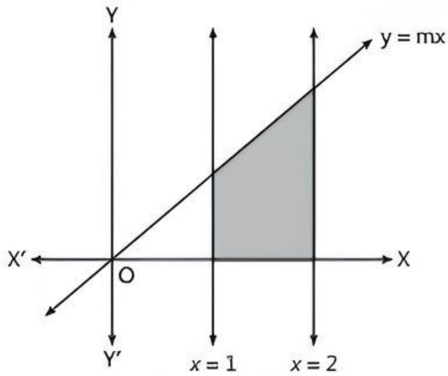
and $y = x$, if $x \geq 0$ ...(2)



$\therefore$ Required area = Area of shaded region

$$= \int_0^5 x\,dx = \left[\frac{x^2}{2}\right]_0^5 = \frac{25}{2} \text{ sq. units}$$

**4.** $y = mx$ is a straight line which passes through origin.




$\therefore$ Required bounded region $= \int_1^2 y\,dx$

$$= \int_1^2 mx\,dx$$

$$= m\left[\frac{x^2}{2}\right]_1^2 = \frac{m}{2}[4-1]$$

$$= \frac{3m}{2} \text{ sq. units}$$

**5.** Equation of parabola; $y^2 = 4ax$ ...(1)

and equation of line; $y = mx$ ...(2)

On solving eqs. (1) and (2), we have

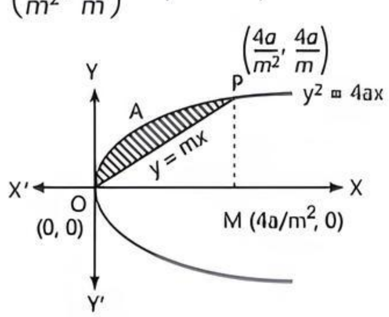
$$m^2x^2 - 4ax = 0$$

$$\Rightarrow x(m^2x - 4a) = 0$$

$$\Rightarrow x = 0, x = \frac{4a}{m^2}$$

$$\therefore y = 0, y = \frac{4a}{m}$$

Thus, the intersection points of parabola and line are $(0, 0)$ and $\left(\frac{4a}{m^2}, \frac{4a}{m}\right)$ respectively.




Let the intersection points of parabola and line be $O$ and $P$ respectively.

Draw a perpendicular on $X$-axis from $P$.

We have to find the area of shaded region OAPO.

Area of region OMPAO $= \int_0^{4a/m^2} y\,dx$

where $y = \sqrt{4ax}$

$$= \int_0^{4a/m^2} \sqrt{4ax}\,dx$$

$$= 2a^{1/2}\int_0^{4a/m^2} x^{1/2}\,dx$$

$$= 2a^{1/2}\left[\frac{x^{3/2}}{3/2}\right]_0^{4a/m^2}$$

$$= 2\sqrt{a} \times \frac{2}{3}\left[\left\{\frac{4a}{m^2}\right\}^{3/2} - 0\right]$$

$$= \frac{4\sqrt{a}}{3} \times \frac{8a\sqrt{a}}{m^3} = \frac{32a^2}{3m^3}$$

**TiP**

*Area of triangle = $\frac{1}{2}$ × base × height*

and area of $\Delta OPM = \frac{1}{2} \times OM \times MP = \frac{1}{2} \times \frac{4a}{m^2} \times \frac{4a}{m}$

$$= \frac{8a^2}{m^3}$$

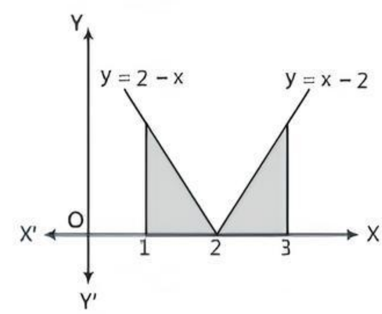
$\therefore$ Required area = Area of region $OMPAO$ − Area of $\Delta OMP$

$$= \frac{32a^2}{3m^3} - \frac{8a^2}{m^3} = \frac{8a^2}{3m^3} \text{ sq. units}$$

**6.** We have, $y = -x + 2 \;\forall\; x < 2$ ...(1)

$y = x - 2 \;\forall\; x \geq 2$ ...(2)

and $x = 1$, $x = 3$




$\therefore$ Required area = area of shaded region

$$= \int_1^2 (2-x)\,dx + \int_2^3 (x-2)\,dx$$

$$= \left[2x - \frac{x^2}{2}\right]_1^2 + \left[\frac{x^2}{2} - 2x\right]_2^3$$

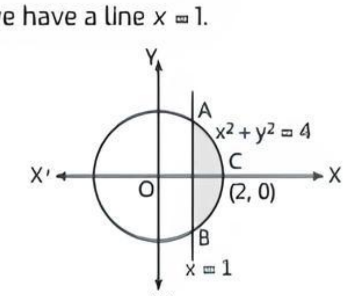
$$= \left[(4-2) - \left(2 - \frac{1}{2}\right)\right] + \left[\left(\frac{9}{2} - \frac{4}{2}\right) - (6-4)\right]$$

$$= \frac{1}{2} + \frac{1}{2} = 1 \text{ sq. unit}$$

**7.** We have, $x^2 + y^2 = 4$ ...(1)

which is a circle with centre (0, 0) and radius = 2

Also we have a line $x = 1$.



$\therefore$ Area of minor segment ABCA $= 2\int_1^2 y\,dx$

$$= 2\int_1^2 \sqrt{4-x^2}\,dx = 2\left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_1^2$$

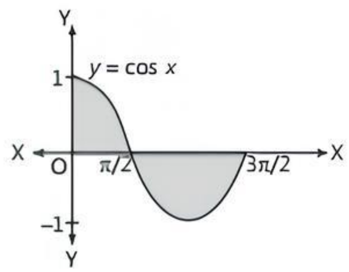
$$= 2\left[0 + 2\sin^{-1}\left(\frac{2}{2}\right) - \left(\frac{1}{2}\sqrt{4-1} + 2\sin^{-1}\frac{1}{2}\right)\right]$$

$$= 2\left[2\times\frac{\pi}{2} - \left(\frac{\sqrt{3}}{2} + 2\times\frac{\pi}{6}\right)\right]$$

$$= 2\left[\pi - \frac{\sqrt{3}}{2} - \frac{\pi}{3}\right] = 2\left[\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right]$$

$$= \left(\frac{4\pi}{3} - \sqrt{3}\right) \text{ sq. units}$$

**8.** We have, $y = \cos x$, whose graph is shown below, between $x = 0$ and $x = \frac{3\pi}{2}$.




$\therefore$ Required area = Area of shaded region

$$= \int_0^{\pi/2} \cos x\,dx + \left|\int_{\pi/2}^{3\pi/2} \cos x\,dx\right|$$

$$= [\sin x]_0^{\pi/2} + \left|[\sin x]_{\pi/2}^{3\pi/2}\right|$$

$$= 1 + |(-1-1)| = 1 + 2 = 3 \text{ sq. units}$$

**9.** We have, $\{(x, y): y^2 \le 2x \text{ and } y \ge x - 4\}$

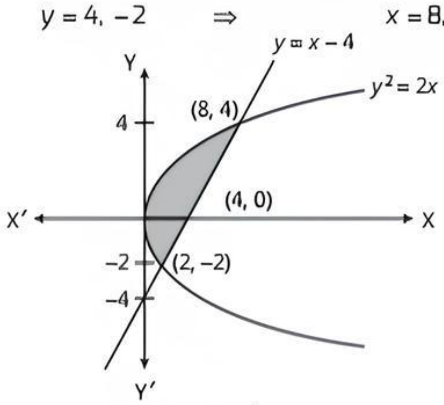
Consider $y^2 = 2x$, which is a parabola open right side and $y = x - 4$ is a straight line which intersect $X$ and $Y$ axes at points $(4, 0)$ and $(0, -4)$ respectively.

The point of intersection of line and parabola is

$$y^2 = 2(y+4)$$

$$\Rightarrow \quad y^2 - 2y - 8 = 0 \Rightarrow (y-4)(y+2) = 0$$

$$\Rightarrow \quad y = 4, -2 \quad \Rightarrow \quad x = 8, 2$$




$\therefore$ Area of shaded region $= \int_{-2}^{4} (x_2 - x_1)\,dy$

$$= \int_{-2}^{4}\left[(y+4) - \left(\frac{y^2}{2}\right)dy\right]$$

$$= \left[\frac{y^2}{2} + 4y - \frac{y^3}{6}\right]_{-2}^{4}$$

$$= \left[\frac{16}{2} + 16 - \frac{64}{6} - \left(\frac{4}{2} - 8 + \frac{8}{6}\right)\right]$$

$$= \left[8 + 16 - \frac{32}{3} - \left(2 - 8 + \frac{4}{3}\right)\right]$$

$$= \left[30 - \frac{36}{3}\right] = 30 - 12 = 18 \text{ sq. units}$$

## Long Answer Type Questions

**1.**

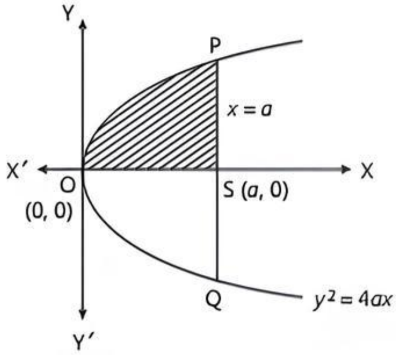
**TiP**

*A chord of a parabola passing through its focus and perpendicular to its axis is known as latus rectum of parabola.*

Equation of parabola: $\quad y^2 = 4ax \quad$ ...(1)

and equation of latus rectum: $x = a \quad$ ...(2)

Let the latus rectum meets the parabola at points P and Q.




We have to find the area of region POQSP which is double the area of shaded region PSOP.

Therefore required area = 2 × area of shaded region

$$= 2\int_0^a y\,dx = 2\int_0^a \sqrt{4ax}\,dx$$

$$= 2\times 2\sqrt{a}\int_0^a x^{1/2}\,dx = 4\sqrt{a}\left[\frac{x^{3/2}}{3/2}\right]_0^a$$

$$= \frac{8}{3}\sqrt{a}\cdot(a^{3/2} - 0) = \frac{8}{3}\sqrt{a}\cdot a\sqrt{a}$$

$$= \frac{8}{3}a^2 \text{ sq. units}$$

**2.** Equation of ellipse: $9x^2 + 16y^2 = 144 \quad$ ...(1)

or $\quad \frac{x^2}{16} + \frac{y^2}{9} = 1 \quad \left[\text{form } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\right]$

For ellipse, semi-major axis $= a = 4$

and minor axis $= b = 3$

Equation of line: $\quad x = 2 \quad$ ...(2)

Clearly, line is parallel to $Y$-axis.

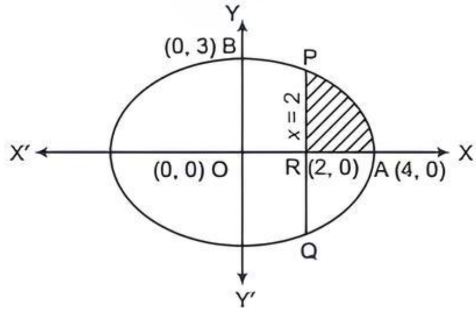
Let line (2), meets the ellipse (1) at point $P$ and $Q$.

Therefore, required area PAQRP

$$= 2 \times \text{Area of shaded part}$$

$$= 2\int_2^4 y\,dx = 2\int_2^4 \frac{1}{4}\sqrt{144 - 9x^2}\,dx \quad \text{[from eq. (1)]}$$

$$= \frac{1}{2}\times 3\int_2^4 \sqrt{16 - x^2}\,dx = \frac{3}{2}\int_2^4 \sqrt{4^2 - x^2}\,dx$$




**TR!CK**

$$\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + C$$

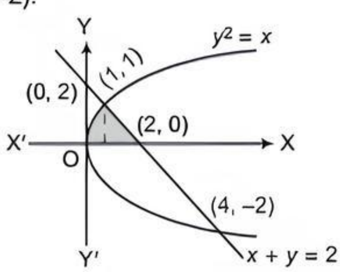
$$= \frac{3}{2}\left[\frac{x}{2}\sqrt{16 - x^2} + \frac{4^2}{2}\sin^{-1}\frac{x}{4}\right]_2^4$$

$$= \frac{3}{2}\left[8\sin^{-1}1 - \left\{1 \times \sqrt{12} + 8\sin^{-1}\frac{1}{2}\right\}\right]$$

$$= \frac{3}{2}\left[8 \times \frac{\pi}{2} - 2\sqrt{3} - 8 \times \frac{\pi}{6}\right]$$

$$= 4\pi - 3\sqrt{3} \text{ sq. units}$$

**3.** The given line and parabola meet at the points (1, 1) and (4, –2).




$\therefore$ Required area = Area of shaded region

$$= \int_0^1 \sqrt{x}\, dx + \int_1^2 (2 - x)\, dx$$

$$= \left[\frac{x^{3/2}}{3/2}\right]_0^1 + \left[2x - \frac{x^2}{2}\right]_1^2$$

$$= \frac{2}{3}(1 - 0) + \left(2 \times 2 - \frac{2^2}{2}\right) - \left(2 - \frac{1}{2}\right)$$

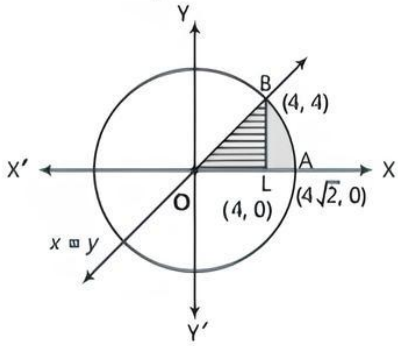
$$= \frac{2}{3} + 2 - \frac{3}{2} = \frac{4 + 12 - 9}{6} = \frac{7}{6} \text{ sq. units}$$

**4.** $x^2 + y^2 = 32$ is a circle whose centre is (0, 0) and radius is $4\sqrt{2}$.

$\therefore$ $\quad y = \sqrt{32 - x^2}$

Now, $\quad x^2 + y^2 = 32 \quad$ ...(1)

and $\quad x = y \quad$ ...(2)




Put the value of $x$ from eq. (2) into eq. (1).

$\therefore \quad y^2 + y^2 = 32$

$\Rightarrow \quad 2y^2 = 32 \Rightarrow y = \pm 4$

$\therefore$ For the first quadrant, $y = 4$.

and $\quad x = 4$

$x = y$ is a line which passes through (0, 0) and (4, 4).

$\therefore$ Required area = Area of OBL + Area of LBA

$$= \int_0^4 x\, dx + \int_4^{4\sqrt{2}} \sqrt{32 - x^2}\, dx$$

**TR!CK**

$$\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + C$$

$$= \left[\frac{x^2}{2}\right]_0^4 + \left[\frac{x\sqrt{32 - x^2}}{2} + \frac{32}{2}\sin^{-1}\frac{x}{4\sqrt{2}}\right]_4^{4\sqrt{2}}$$

$$= \left(\frac{16}{2} - 0\right) + \left[\left(0 + 16\sin^{-1}1 - \left(8 + 16\sin^{-1}\frac{1}{\sqrt{2}}\right)\right)\right]$$

$$= 8 + 16 \cdot \frac{\pi}{2} - 8 - 16 \cdot \frac{\pi}{4}$$

$$= 8\pi - 4\pi = 4\pi \text{ sq. units}$$

**COMMON ERR!R**

*Sometimes students takes wrong limits of common region and get the wrong value of area.*

*e.g.,* $\int_0^{4\sqrt{2}} x\, dx$ *(this is wrong)*

*Divide the shaded region in two parts where the curves intersect and take the right values of limits of both parts.*

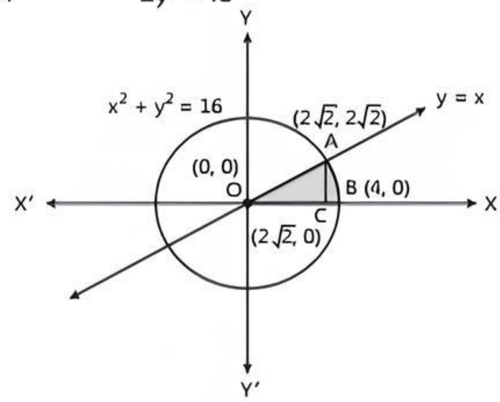
**5.** The area bounded by the circle $x^2 + y^2 = 16$, $y = x$ and X-axis is the area OABO.

On solving $x^2 + y^2 = 16$ and $y = x$, we have

$(y)^2 + y^2 = 16$

$\Rightarrow \quad y^2 + y^2 = 16$

$\Rightarrow \quad 2y^2 = 16$




$\Rightarrow \quad y^2 = 8$

$\Rightarrow \quad y = 2\sqrt{2}$

[In the first quadrant, $y$ is positive]

When $y = 2\sqrt{2}$, then $x = 2\sqrt{2}$

So, the point of intersection of the given line and circle in the first quadrant is $(2\sqrt{2}, 2\sqrt{2})$.

$\therefore$ Required area = Area of the shaded region OABO

= Area of OACO + Area of ABCA

$= \frac{1}{2} \times OC \times AC + \int_{2\sqrt{2}}^{4} y\,dx$

$= \frac{1}{2} \times 2\sqrt{2} \times 2\sqrt{2} + \int_{2\sqrt{2}}^{4} \sqrt{16 - x^2}\,dx$

$= 4 + \left[\frac{x}{2}\sqrt{16 - x^2} + \frac{16}{2}\sin^{-1}\frac{x}{4}\right]_{2\sqrt{2}}^{4}$

$= 4 + \left[0 + 8\sin^{-1}\left(\frac{4}{4}\right) - \left(\frac{2\sqrt{2}}{2}\sqrt{16-8} + 8\sin^{-1}\frac{2\sqrt{2}}{4}\right)\right]$

$= 4 + 8\sin^{-1}(1) - \left(\sqrt{2} \times 2\sqrt{2} + 8\sin^{-1}\frac{1}{\sqrt{2}}\right)$

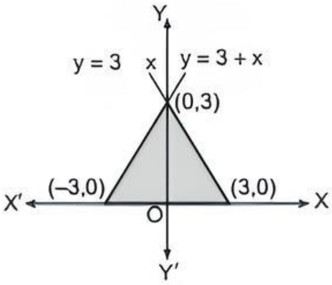
$= 4 + 8 \times \frac{\pi}{2} - \left(4 + 8 \times \frac{\pi}{4}\right)$

$= 4\pi - 2\pi = 2\pi$ sq. units

**6.** We have, $y = 3 - |x|$

$\Rightarrow \quad y = 3 + x, \forall\, x < 0 \quad$ ...(1)

and $\quad y = 3 - x, \forall\, x \geq 0 \quad$ ...(2)




$\therefore$ Required area = Area of shaded region

$= 2\int_{-3}^{0} (3 + x)\,dx$

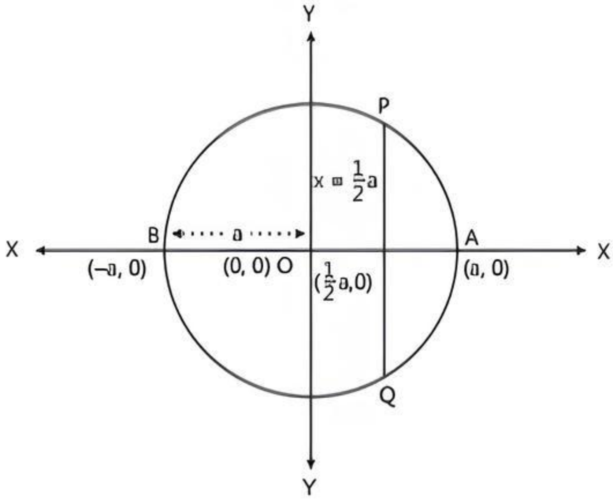
$= 2\left[3x + \frac{x^2}{2}\right]_{-3}^{0}$

$= -2\left[-9 + \frac{9}{2}\right]$

$= -2 \times \frac{-9}{2} = 9$ sq. units

**7.** Let line $x = \frac{1}{2}a$, intersects the circle $x^2 + y^2 = a^2$ at points $P$ and $Q$.

Thus the complete circle is divided into two parts PAQP and PBQP by the line PQ.




$\therefore$ Area of smaller part PAQP $= 2\int_{a/2}^{a} y\,dx$

$= 2\int_{a/2}^{a} \sqrt{a^2 - x^2}\,dx \qquad [\because x^2 + y^2 = a^2]$

**TiP**

*Learn to apply limits correctly to avoid errors.*

$= 2\left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_{a/2}^{a}$

$= 2\left[\frac{a}{2}\sqrt{a^2 - a^2} + \frac{a^2}{2}\sin^{-1}\frac{a}{a}\right]$

$-2\left[\frac{a}{4}\cdot\sqrt{a^2 - \frac{1}{4}a^2} + \frac{a^2}{2}\sin^{-1}\frac{\frac{a}{2}}{a}\right]$

$= 2\left[0 + \frac{a^2}{2}\sin^{-1}1\right] - 2\left[\frac{a}{4}\cdot\frac{1}{2}\cdot a\sqrt{3} + \frac{a^2}{2}\sin^{-1}\frac{1}{2}\right]$

$= a^2 \cdot \frac{\pi}{2} - \frac{1}{4}a^2\sqrt{3} - a^2 \cdot \frac{\pi}{6}$

$= \frac{\pi a^2}{3} - \frac{a^2\sqrt{3}}{4} = \frac{a^2}{12}(4\pi - 3\sqrt{3})$ sq. units

$\therefore$ Area of larger part PBQP = Area of complete circle − Area of smaller part PAQP

$= \pi a^2 - \frac{a^2}{12}(4\pi - 3\sqrt{3}) = \frac{a^2}{12}(12\pi - 4\pi + 3\sqrt{3})$

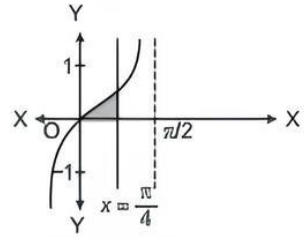
$= \frac{a^2}{12}(8\pi + 3\sqrt{3})$ sq. units

Therefore, required ratio

$= \frac{a^2}{12}(8\pi + 3\sqrt{3}) : \frac{a^2}{12}(4\pi - 3\sqrt{3})$

$= (8\pi + 3\sqrt{3}) : (4\pi - 3\sqrt{3})$

**8.** We have, $y = \tan x$ and $x = \frac{\pi}{4}$




$\therefore$ Required area = Area of shaded region

$= \int_{0}^{\pi/4} \tan x\,dx$

$= [-\log|\cos x|]_0^{\pi/4}$

$= -\log\frac{1}{\sqrt{2}} + \log 1$

$= \log\sqrt{2} = \frac{1}{2}\log 2$ sq. units

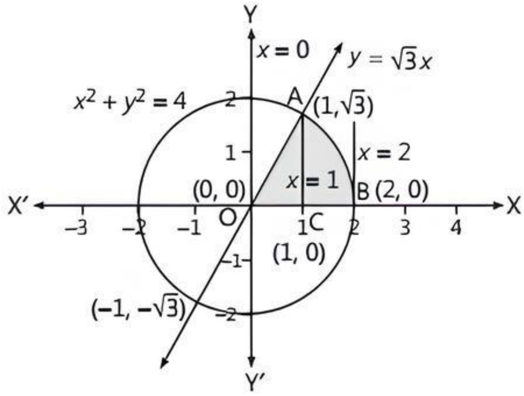
**9.** The area bounded by the circle $x^2 + y^2 = 4$, line $y = \sqrt{3}x$ and $X$-axis is the area OABCO.

On solving $x^2 + y^2 = 4$ and $y = \sqrt{3}x$, we have

$x^2 + (\sqrt{3}x)^2 = 4$

$\Rightarrow \quad x^2 + 3x^2 = 4$

$\Rightarrow \quad 4x^2 = 4 \Rightarrow x = 1$ (In the first quadrant)




When $x = 1$, then $y = \sqrt{3}$.

So, the point of intersection of the given line and circle in the first quadrant is $(1, \sqrt{3})$.

$\therefore$ Required area = Area of the shaded region OABCO

$= \text{Area of OACO} + \text{area of ABCA}$

$= \frac{1}{2} \times OC \times AC + \int_1^2 y\, dx$

$= \frac{1}{2} \times 1 \times \sqrt{3} + \int_1^2 \sqrt{4 - x^2}\, dx$

$= \frac{\sqrt{3}}{2} + \left[\frac{x}{2}\sqrt{4 - x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_1^2$

$= \frac{\sqrt{3}}{2} + \left[0 + 2 \cdot \frac{\pi}{2} - \frac{1}{2} \cdot \sqrt{3} - 2 \cdot \frac{\pi}{6}\right]$

$= \frac{\sqrt{3}}{2} + \pi - \frac{\sqrt{3}}{2} - \frac{\pi}{3}$

$= \frac{2\pi}{3}$ sq. units

**10.** We have, $\{(x, y) : 0 \le y \le x^2 + 1, 0 \le y \le x + 1, 0 \le x \le 2\}$

Here consider $y = x^2 + 1$, which is an equation of parabola open upwards with vertex (0, 1).

Also, $y = x + 1$ which is an equation of line, intercept $X$ and $Y$ axes at $-1$ and 1 respectively.

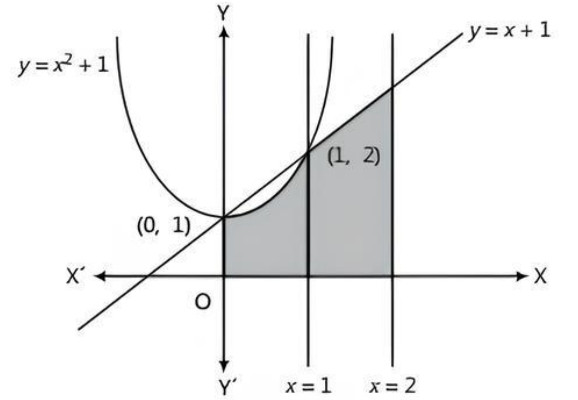
On solving above equations,

$x^2 + 1 = x + 1 \Rightarrow x^2 - x = 0$

$\Rightarrow \quad x = 0, 1$

$\Rightarrow \quad y = 1, 2$

So, point of intersections are (0, 1) and (1, 2).




$\therefore$ Required area $= \int_0^1 y\, dx + \int_1^2 y\, dx$

$= \int_0^1 (x^2 + 1)\, dx + \int_1^2 (x + 1)\, dx$

$= \left[\frac{x^3}{3} + x\right]_0^1 + \left[\frac{x^2}{2} + x\right]_1^2$

$= \left[\frac{1}{3} + 1 - 0\right] + \left[\frac{4}{2} + 2 - \left(\frac{1}{2} + 1\right)\right]$

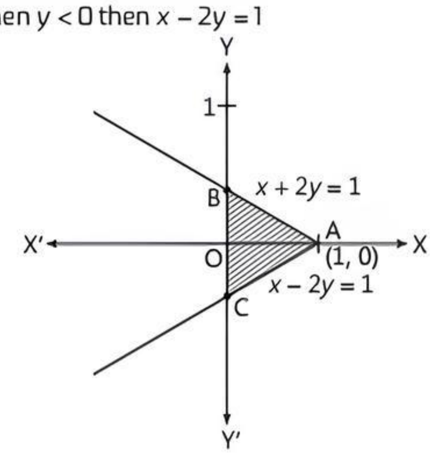
$= \frac{4}{3} + 4 - \frac{3}{2} = \frac{8 + 24 - 9}{6} = \frac{23}{6}$ sq units

**11.** Given curves are $x = 0$ and $x + 2|y| = 1$.

Now, $\quad x + 2|y| = 0$

When $y > 0$, then $x + 2y = 1$

When $y < 0$ then $x - 2y = 1$




$\therefore$ Area of bounded region $ABC$

$= 2 \times \text{area of } AOB = 2\int_0^1 \left(\frac{1 - x}{2}\right) dx$

$= \left[x - \frac{x^2}{2}\right]_0^1 = 1 - \frac{1}{2} - (0 - 0) = \frac{1}{2}$ sq. unit

# Chapter Test

## Multiple Choice Questions

**Q 1. The area of region bounded by the line $y = 8x$, the $X$-axis and the lines $x = 1$ and $x = 2$ is:**

a. 11 sq. units
b. 10 sq. units
c. 12 sq. units
d. 5 sq. units

**Q 2. The area of the region bounded by the curve $y = x^2$ and the line $y = 16$ is:**

a. $\frac{32}{3}$ sq. units
b. $\frac{256}{3}$ sq. units
c. $\frac{64}{3}$ sq. units
d. $\frac{128}{3}$ sq. units

## Assertion and Reason Type Questions

**Directions (Q. Nos. 3-4):** *In the following questions, each question contains Assertion (A) and Reason (R). Each question has 4 choices (a), (b), (c) and (d) out of which only one is correct. The choices are:*

a. Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of Assertion (A)
b. Both Assertion (A) and Reason (R) are true but Reason (R) is not the correct explanation of Assertion (A)
c. Assertion (A) is true and Reason (R) is false
d. Assertion (A) is false and Reason (R) is true

**Q 3.** Assertion (A): The area bounded by the line $y = 2x$, the $X$-axis and the lines $x = -2$ and $x = 2$ is 8 sq. units.

Reason (R): If for $x \in [a, c]$, $f(x) \geq 0$ and for $x \in [c, b]$, $f(x) \leq 0$, where $a < c < b$, then area of region bounded by curve $y = f(x)$, $X$-axis, $x = a$ and $x = b$ is given by

$$\text{Area} = \int_a^a f(x)\,dx - \int_c^b f(x)\,dx$$

**Q 4.** Assertion (A): The area of region bounded by the curve $y = |x|$, $x = -1$, $x = 2$ and $X$-axis is 5 sq. units.

Reason (R): Area of the region bounded by the curve $y^2 = 2x$ the $Y$-axis and the line $y = 2$ is $\frac{4}{3}$ sq. units.

## Case Study Based Questions

**Q 5. Case Study 1**

A mirror in the shape of an ellipse is represented by $\frac{x^2}{4} + \frac{y^2}{9} = 1$ was hanging on the wall. Anika and her sister were playing with football inside the house, even her mother refused to do so. All of a sudden, football hit the mirror and got a scratch in the shape of line represented by $x = \sqrt{3}$.

*Based on the above information, solve the following questions:*

(i) Find the point(s) of intersection of ellipse (mirror) and scratch (straight line).
(ii) Find the area of ellipse (mirror).
(iii) Find the area of small part of the ellipse (mirror) divided by scratch (straight line).

*Or*

Evaluate $\int_{\sqrt{2}}^{\sqrt{3}} \sqrt{4 - x^2}\,dx$.

**Q 6. Case Study 2**

Suppose $f$ is an absolute function defined from $f: R \to R$ such that $f(x) = \left| x - \frac{1}{2} \right|$,

where $\left| x - \frac{1}{2} \right| = \begin{cases} x - \frac{1}{2}, & x \geq \frac{1}{2} \\ \frac{1}{2} - x, & x < \frac{1}{2} \end{cases}$

*Based on the above information, solve the following questions:*

(i) Find the area between the curve $f(x) = \left| x - \frac{1}{2} \right|$, $X$-axis and the lines $x = 1$ and $x = 3$.
(ii) Find the area of triangle formed by the curve and between axes.
(iii) Find the area of region bounded by the curve $f(x)$ and the lines $y = \frac{1}{8}$ to $y = \frac{1}{4}$.

*Or*

Find the area of unshaded region of triangle formed by the curve and between axes.

## Very Short Answer Type Questions

**Q 7.** Find the area lying in the first quadrant bounded by the circle $x^2 + y^2 = 16$ and the lines $x = 0$, $x = 4$.

**Q 8.** Find the area of the parabola $y^2 = 4ax$ bounded by its latus rectum in first quadrant.

## Short Answer Type-I Questions

**Q 9.** Find the area bounded by the curve $y = \frac{1}{2}x^2$, the $X$-axis and the ordinate $x = 2$.

**Q 10.** Draw the graph of the curve $y = |\sin x|$ and find the area bounded by the curve, $X$-axis and ordinates $x = -\pi$ to $2\pi$.

## Short Answer Type-II Questions

**Q 11.** Sketch the graph of $y = |x + 3|$ and evaluate $\int_{-6}^{0} |x + 3|\,dx$.

**Q 12.** Find the area bounded by the curve $y = \cot x$, $X$-axis and the lines $x = \frac{\pi}{2}$ to $x = \frac{3\pi}{4}$.

## Long Answer Type Questions

**Q 13.** Find the area bounded between the curve $y^2 = 4x$, line $x + y = 3$ and $Y$-axis.

**Q 14.** Find the area between $X$-axis, curve $x = y^2$ and its normal $y + 2x = 3$ at the point (1, 1).